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A SIMPLE, PRACTICAL METHOD FOR THE EXPERIMENTAL DETERMINATION OF THE END FIXITY OF A COLUMN

By

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September 1970

U. S. ARMY AVIATION MATERIEL LABORATORIES

FORT EUSTIS, VIRGINIA

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STANFORD UNIVERSITY

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This program was carried out under Contract DAAJ02-68-C-0035 with Stanford University as prime contractor and Georgia Institute of Technology as subcontractor.

This research was directed toward a critical study of column end fixity. A simple method for experimentally determining the end fixity of a column is presented.

This report has been reviewed by the U.S. Army Aviation Materiel Laboratories and is considered to be technically sound. It is published for the exchange of information and the stimulation of future research.

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U. S. ARMY AVIATION MATERIEL LABORATORIES
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SUMMARY

The question of column end fixity is critically reconsidered. An effective method for determination is established. This method relies upon the fact that there exists a readily determined point in the span of the column for which the deflection due to a given side force is maximum. The product of the coefficient which defines the flexibility at this point and the Euler load appropriate to the actual end conditions is, to all intents and purposes, constant.

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LIST OF SYMBOLS

A_m	coefficient in deflection function expansion, in.
a	beam taper ratio
E	modulus of elasticity, lb/in.^{2}
f	$= \delta/Q$ = flexibility coefficient, in./lb
f_1	flexibility coefficient for beam with two halves like OA (Figure 1a), in./lb
f_2	flexibility coefficient for beam with two halves like OA' (Figure 1a), in./lb
$\tilde{f}_2$	total flexibility coefficient, as defined in Equation (43), in./lb
f_3	total flexibility coefficient, as defined in Equation (49), in./lb
f_n	total flexibility coefficient, as defined in Equation (51), in./lb
I	beam cross-section moment of inertia, in.^{4}
I_o	beam cross-section moment of inertia for $\xi = o$, in.^{4}
i	dummy index
k	spring stiffness, lb/in. for lateral spring, or lb-in. for rotational spring
L	column length, in.
L_1	distance to load application point when the product $P_{cr}f$ is exact, in.
M_o	moment at $\xi = 0$, in.-lb
M_o	moment due to dummy load, in.
M_1	actual moment in beam, in.-lb
m	dummy index of summation

LIST OF SYMBOLS - CONT'D

N	number of terms in deflection function expansion
n	number of concentrated loads applied in intermediate lateral support cases
P_{cr}	column buckling load, lb
$_1P_{cr}$	buckling load for column with two halves like OA (Figure 1A), lb
$_2P_{cr}$	buckling load for column with two halves like OA' Figure 1a), lb
Q	magnitude of concentrated lateral force, lb
R	vertical reaction at $\xi = 1$
U	strain energy, in.-lb
U^*	internal complementary work, in.-lb
u	$= \frac{L}{2}\sqrt{\frac{P_{cr}}{EI}}$ = buckling coefficient
W	beam lateral deflection function, in.
x	beam axial coordinate, in.
α	$= \frac{P_{cr}f}{\pi^2 L/48}$ = error factor
δ	beam deflection at the point of load application, in.
δ_1	beam deflection at the point of the first load application
δ_2	beam deflection at the point of the second load application
δ_3	beam deflection at the point of the third load application
δ_i	beam deflection at the point of the i^{th} load application
θ	beam slope at $\xi = 0$
λ	load location parameter

LIST OF SYMBOLS - CONT'D

ξ = x/L = dimensionless beam axial coordinate

Π total potential energy, in.-lb

Π^* total complementary potential, in.-lb

$\phi(u)$ function defined in Equation (27)

$\psi(u)$ function defined in Equation (27)

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INTRODUCTION

Discrepansies between the conditions assumed in theory and those achieved in the laboratory or in practical application seriously influence the agreement between theoretical predictions and actual observations on the stability of structures. Admittedly, methods of data interpretation which enable valid deductions with regard to the behavior of perfect or ideal structures to be made from information obtained from tests on imperfect or realistic structures have been developed.[1,2,3] Nevertheless, there are still some aspects of the question that merit further review and consideration. Prime areas, in this regard, are to be found in the subjects of boundary restraint and in the development of nondestructive testing methods that can give information with regard to actual behavior under service conditions.

The subject of boundary restraint has long been recognized for the simple column. Indeed, since the very earliest tests,[4] experimentalists have, as the literature amply illustrates,[5,6] striven to develop end fixtures or restraint devices that would approximate as closely as possible the boundary conditions that the analyst can treat. That they have, in many cases, closely reached these ideal conditions is borne out by the experimental load values that they have achieved. However, it is equally clear that in no case do these special devices even begin to approximate the conditions met in real structures. We must admit that theoretical and experimentally achieved boundary conditions are often very much apart from conditions of realistic restraint.

The problem is seriously magnified when the instability of a plate is examined, for even if the plate is uniform, homogeneous, and isotropic, considerable difficulty is experienced. This viewpoint is readily confirmed by a review of the current literature on experimental studies on plates. When the subject of sandwich plates is considered, the difficulties are compounded. The conditions of edge restraint are much more complex for structures of this type than for thin plates, a point clearly made by Benson and Mayers.[7]

With regard to the subject of shells, the matter is no less important; however, it has received attention only in more recent times. The majority of the work in this area is analytical, but there are several important contributions.[8,9,10,11 12] In stressing the importance of the subject with regard to this class of structure, the earlier recognition of the problem by Love and Southwell[13,14] cannot be ignored.

In a previous study[5] the various attempts that the experimentalist has made to duplicate, in the laboratory, the conditions prescribed by the analyst have been reviewed. This work was not intended to extend or refine these studies. A different approach was followed. From the outset, the viewpoint has been that the designer creates that which has never been; the analyst investigates that which has been devised. Thus, the axiom on which the study was based is that the prime purpose of analysts must be to devise methods of predicting with reliability the behavior of systems which

are subject to the constraints of reality. Mathematical analysis and experimental study are considered valid only insofar as they lead to evaluations in this sense; in any other fashion they are purely academic exercises. In the literature that exists on the influence of boundary conditions on both stability and strength of structures, this point is frequently lost sight of. All too often the theoretician gives no indication of how his prescribed boundary conditions might be achieved in practice or the degree of error resulting from deviations. On the other hand, the experimentalist usually tends to concentrate, not on determining the parameters which define the achievable boundaries, but rather on attempting to reproduce the conditions which the theoretician has specified. The practicing engineer, the designer, requires methods, experimental and theoretical, of obtaining a relationship between easily determined parameters and actual restraint. In this report, we establish a rationale.

DISCUSSION OF THEORY

We begin this study by asking what is the influence of boundary restraint on the deflection and/or stability of a simple member under various loading conditions?

We note, in the case of a column under axial compression, that when this member is pinned at both ends, the critical load level is given by the expression

$$P_{cr} = \frac{\pi^2 EI}{L^2} \tag{1}$$

provided the strut is reasonably slender. If the end fixity condition is changed from pinned to fixed, then the load level is substantially altered. It becomes

$$P_{cr} = \frac{4\pi^2 EI}{L^2} \tag{2}$$

Now, consider the same simple member, but instead of applying a destabilizing compressive force, apply a lateral force at its midpoint. There is no critical load level for this force system from a stability viewpoint, but there is, of course, a critical stress. So long as the load is not approaching the level that would induce this stress, the deflection is linear with load and, for the two end fixity cases considered, is given by

$$\delta = \frac{QL^3}{48\ EI} \quad \text{(pinned ends)} \tag{3}$$

$$\delta = \frac{QL^3}{192\ EI} \quad \text{(fixed ends)} \tag{4}$$

It is clear from these readily established relationships that the change in end fixity from pinned to fixed causes an increase in critical load level and a decrease in deflection for a given force. It is interesting to note that the destabilizing force grows by a factor of four; the deflection decreases to one-fourth. Consequently, for these two configurations, the identity

$$P_{cr} \cdot \frac{\delta}{Q} = \frac{\pi^2 L}{48} \tag{5}$$

is established.

Now the quantity $\frac{\delta}{Q}$ is in essence a flexibility coefficient, and so we may say for these two cases, at any rate, that the product of the critical load for a centrally loaded slender column and the flexibility coefficient for a central lateral force acting on said column is dependent only upon the length of the column. Symbolically:

$$P_{cr} f = \frac{\pi^2 L}{48} = \text{Constant} \tag{6}$$

The importance of this relationship, if it can be extended to other cases, is clear. It provides the simplest possible method of determining critical load from easily obtained quantities.

It is important to note that the two examples given so far have certain features in common:

1. They are for geometrically uniform bodies.

2. They are for bodies with symmetry and ideal end support.

To extend the concept further, we begin by systematically removing these conditions.

In the first case, the restriction to nonuniform bodies is removed; in subsequent sections, the support conditions are varied to reflect realistic restraint, including intermediate support.

NONUNIFORM COLUMNS

To begin, we consider the strut depicted in Figure 1a. This strut is non-uniform, the two halves having different cross-sectional areas and moments thereof.

The critical load for a strut that is unsymmetrical about the center has been investigated by Case,[15] who has shown that the value of P_{cr} is given by the expression

$$\frac{2}{P_{cr}} = \frac{1}{{}_1P_{cr}} + \frac{1}{{}_2P_{cr}} \tag{7}$$

where ${}_1P_{cr}$ = the crippling load of a strut with two halves like OA, and ${}_2P_{cr}$ = the crippling load of a strut with two halves like OA$'$. It is but a matter of simple calculation to show that if f_1 is the flexibility coefficient for a centrally loaded beam of total length L and of the form of OA, and if f_2 is the value of this parameter corresponding to a beam entirely like OA$'$, then the flexibility coefficient for the composite beam is given by

$$2f = f_1 + f_2 \tag{8}$$

It follows from equations (7) and (8) that

$$P_{cr}f = \frac{{}_1P_{cr} \cdot {}_2P_{cr}\,(f_1 + f_2)}{{}_1P_{cr} + {}_2P_{cr}} \tag{9}$$

But

$${}_1P_{cr}f_1 = {}_2P_{cr}f_2 = \frac{\pi^2 L}{48} \tag{10}$$

Hence,

$$P_{cr}f = \frac{\pi^2 L}{48} \tag{11}$$

It is apparent from this that the condition of complete uniformity of geometry is not essential. We conclude, too, that the point of load application is not, therefore, the point of symmetry but rather the point of maximum compliance. To extend the argument, we examine next the case of a column with continuously varying section geometry. We shall assume (for convenience) that the variation in EI is described by the expression

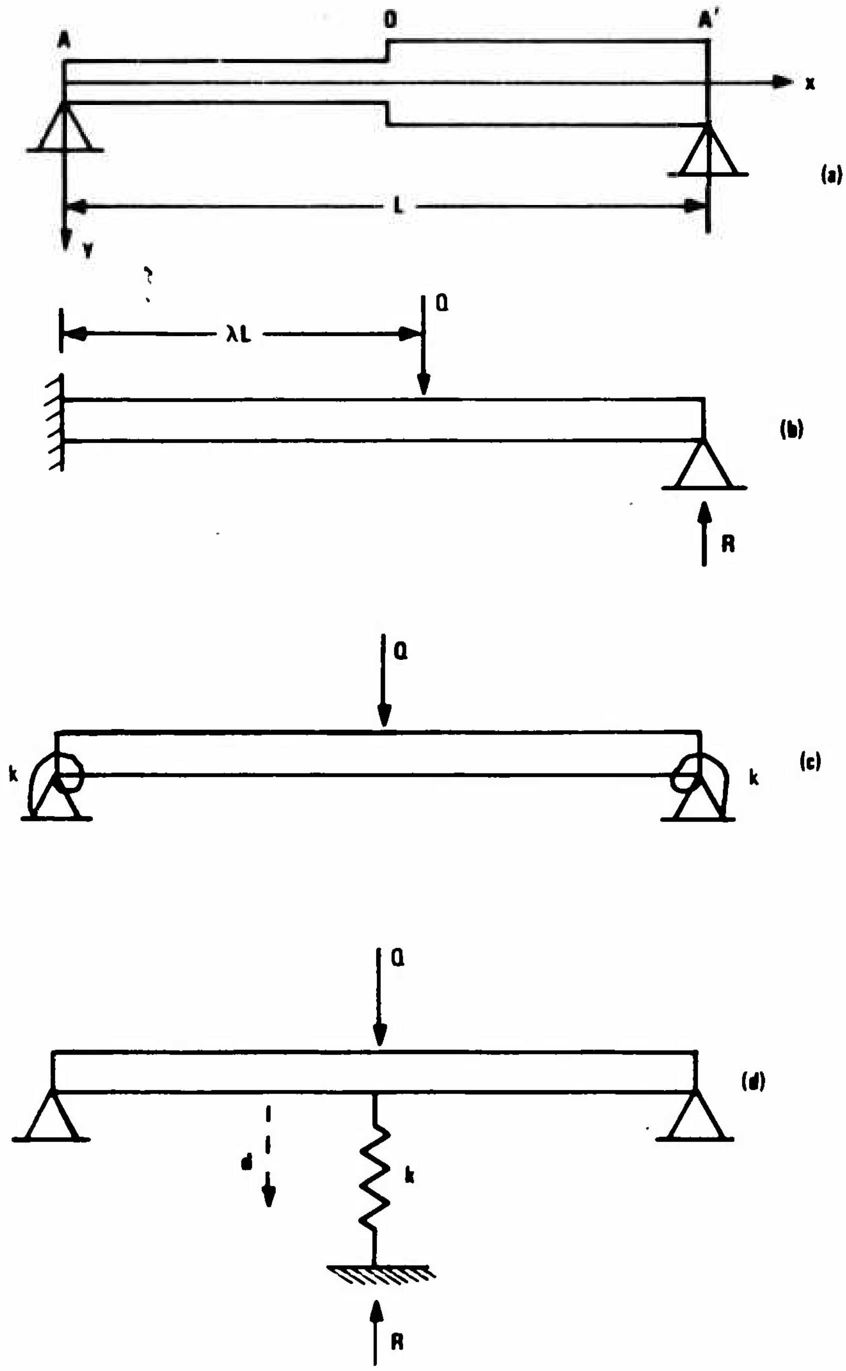

Figure 1. Diagrams of Several of the Beam Configurations Studied.

$$EI = EI_o \left[1 - \frac{1}{3} \left(\frac{x}{L}\right)^2 \right] \tag{12}$$

and that the end $x = 0$ is built in while the end $x = L$ is pinned. It can be shown[16] that this system has a critical load that is given by

$$17.79 \frac{EI_o}{L^2} \leq P_{cr} \leq 17.88 \frac{EI_o}{L^2} \tag{13}$$

This strut has a 33% variation in the value of EI as we proceed from one end to the other, a variation that is greater than one might expect in a practical application.

The flexibility coefficient under lateral force can be computed easily from energy considerations (Appendix I). There results

$$f = 0.0109 \frac{L^3}{EI_o} \tag{14}$$

and so

$$P_{cr} f = 0.945 \frac{\pi^2 L}{48} \tag{15}$$

The error between this result and that established in equation (6) amounts to less than 6%, which, when viewed from a practical point, is excellent.

NONUNIFORM SUPPORT

The simple examples given have provided substantial evidence that a practical method for determination of end fixity in columns exists and that its application would provide a nondestructive process for determining, in the field, those factors that are not readily amenable to calculation. But there is some weakness in the argument, since all the results presented apply to the most mathematically ideal boundary conditions - conditions that are similar at both ends of the member, a situation unlikely in practice. There is lack of realism, too, in the sense that we have not considered the variations in rotational and translational end restraint which characterize virtually every frame structure. The analysis is, therefore, extended to cover these situations.

To begin, we consider the case of a uniform member encastre at one end and pinned at the other (Figure 1b). This is a classical problem and the various parameters are well established. The critical compressive load is given by

$$P_{cr} = 2.05\, \pi^2 \frac{EI}{L^2} \qquad (16)$$

The flexibility coefficent at the point of maximum compliance (x = 0.586) is given by

$$f = 0.00984\, \frac{L^3}{EI} \qquad (17)$$

and, thus, it follows that the product

$$P_{cr} f = 0.968\, \frac{\pi^2 L}{48} \qquad (18)$$

This is not precisely of the form of equation (6), but the 4% variation from that value is, nevertheless, small.

Next, we consider the situation shown in Figure 1c of a pinned uniform column but with equal rotational restraint at the ends. The critical compressive load for this case is determined as a function of spring stiffness from the transcendental equation:

$$- \frac{2EI}{kL} = \frac{\tan u}{u} \qquad (19)$$

where

$$P_{cr} = \frac{4u^2 EI}{L^2} \qquad (20)$$

It is obvious from the support symmetry that the point of minimum lateral stiffness is the midpoint of the beam. The flexibility coefficient, f, is easily determined, in closed form, for this case (see Appendix II) and is

$$f = \frac{L^3}{48EI} \left[1 - \frac{3}{4(1 + \frac{2EI}{kL})} \right] \qquad (21)$$

The product $P_{cr}f$ can be written

$$P_{cr}f = \frac{4u^2}{\pi^2} \left[1 - \frac{3}{4(1 + \frac{2EI}{kL})} \right] \frac{\pi^2 L}{48} \qquad (22)$$

It appears from this result that $P_{cr}f$ is no longer a constant for variations in support but, instead, by reference to equation ([illegible]), is in error by a factor

$$\alpha = \frac{4u^2}{\pi^2} \left[1 - \frac{3}{4(1 + \frac{2EI}{kL})} \right] \frac{\pi^2 L}{48} \qquad (23)$$

When the factor α is plotted against spring stiffness, k (Figure [illegible]), it is clear that the variation from unity is small and that the maximum error is less than 12% over all values of end fixity.

These results can be systematically extended to cases of nonsymmetric end restraint. Suppose the previous example is taken but with one spring removed. The critical load for this configuration is determined from the transcendental equation

$$\frac{EI}{kL} = \frac{-1}{2u} \left(\frac{1}{2u} - \frac{1}{\tan 2u} \right) \qquad (24)$$

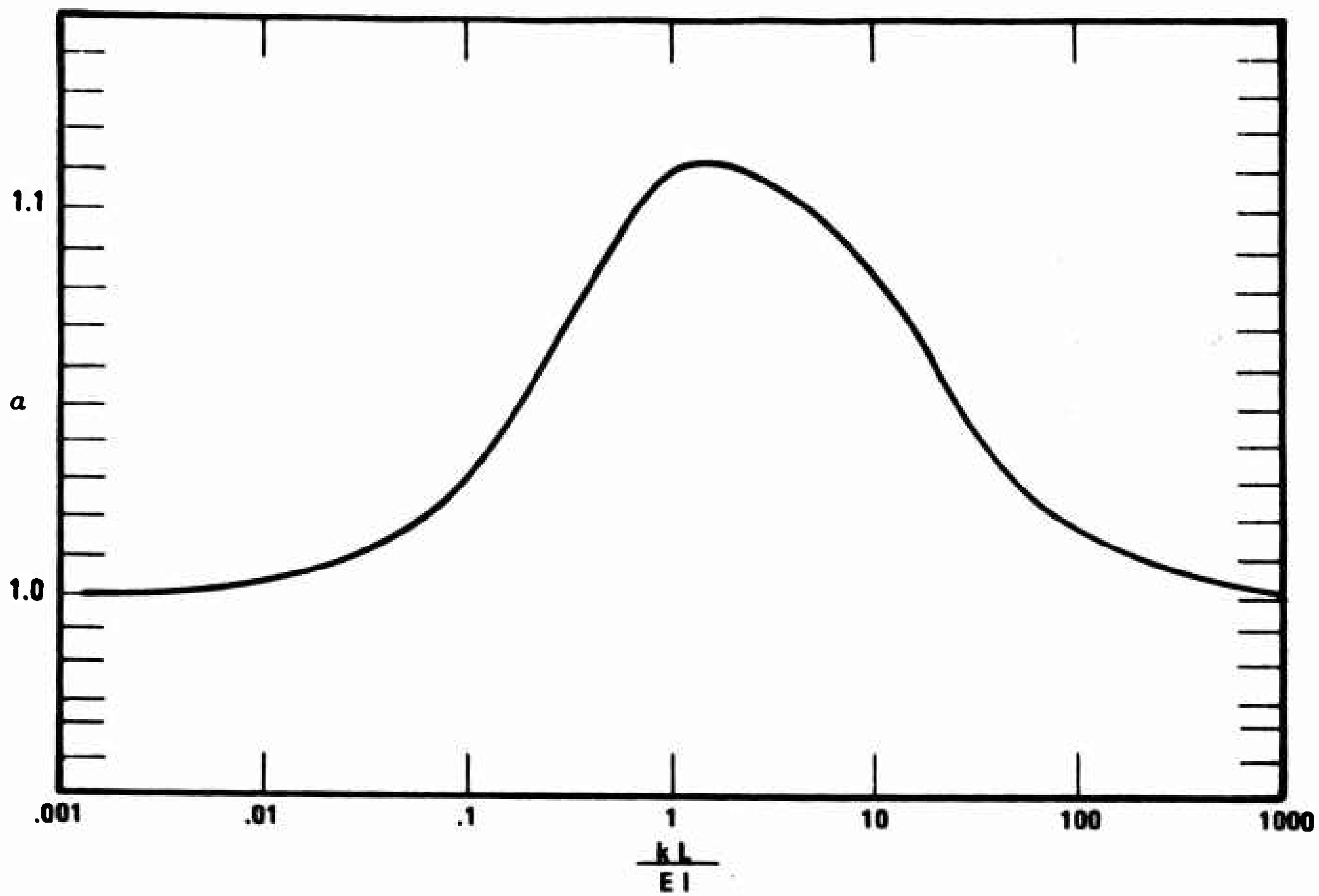

Figure 2. A Plot of the Error Factor, α, Against the Rotational Restraint at Both Ends of a Pinned Beam (Figure 1c).

where again

$$P_{cr} = 4\,u^2 \frac{EI}{L^2} \tag{25}$$

The flexibility coefficient must now be determined for given values of end restraint. The analysis is somewhat more involved because the point of maximum value (minimum lateral stiffness) depends on end restraint. A summary of the procedure used here is given in Appendix III. The product of $P_{cr}f$ and the maximum flexibility coefficient is plotted versus end restraint in Figure 3. It is readily apparent here that the product is approximately constant for unsymmetric changes in end restraint and is directly proportional to column length. Again, the constant is

$$\frac{\pi^2 L}{48}$$

and the maximum error is less than 4% for all values of restraint. It is important from a practical point to note here that the locus of points of minimum stiffness shown in Figure 3 does not deviate from the column mid-point by more than 10%.

This example can be continued by taking the above spring stiffness as infinite and then adding degrees of rotational restraint to the opposite end. The critical load for this case is determined from the equation

$$\varphi^2(u) = 4\psi(u)\left[\frac{3EI}{kL} + \psi(u)\right] \tag{26}$$

where

$$\varphi(u) = \frac{3}{u}\left[\frac{1}{\sin 2u} - \frac{1}{2u}\right] \text{ and } \psi(u) = \frac{3}{2u}\left[\frac{1}{2u} + \frac{1}{\tan 2u}\right] \tag{27}$$

and

$$P_{cr} = 4u^2 \frac{EI}{L^2} \tag{28}$$

Again the flexibility coefficient and its point of minimum value depend on end restraint. When the product $P_{cr}f$ is plotted against restraint, Figure 4, it is clear as before that the result is approximately constant throughout the complete range of restraint. The maximum deviation from $\pi^2 L/48$ is less than 4%, and the locus of points of minimum lateral stiffness is within 10% of the column midpoint.

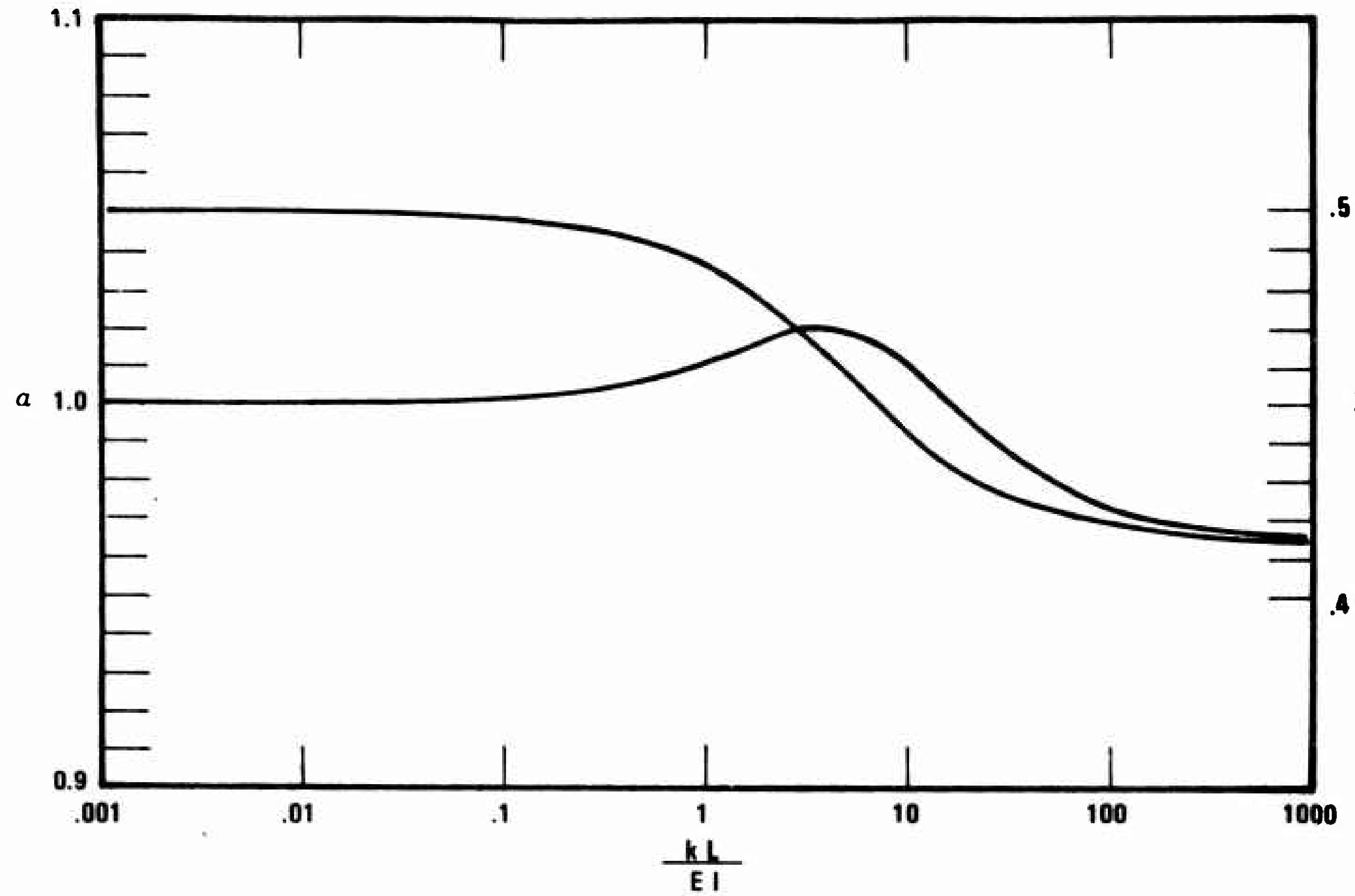

Figure 3. A Plot of the Error Factor, α, and Maximum Flexibility Point, λ, Against Rotational End Restraint at One End of a Pinned Beam.

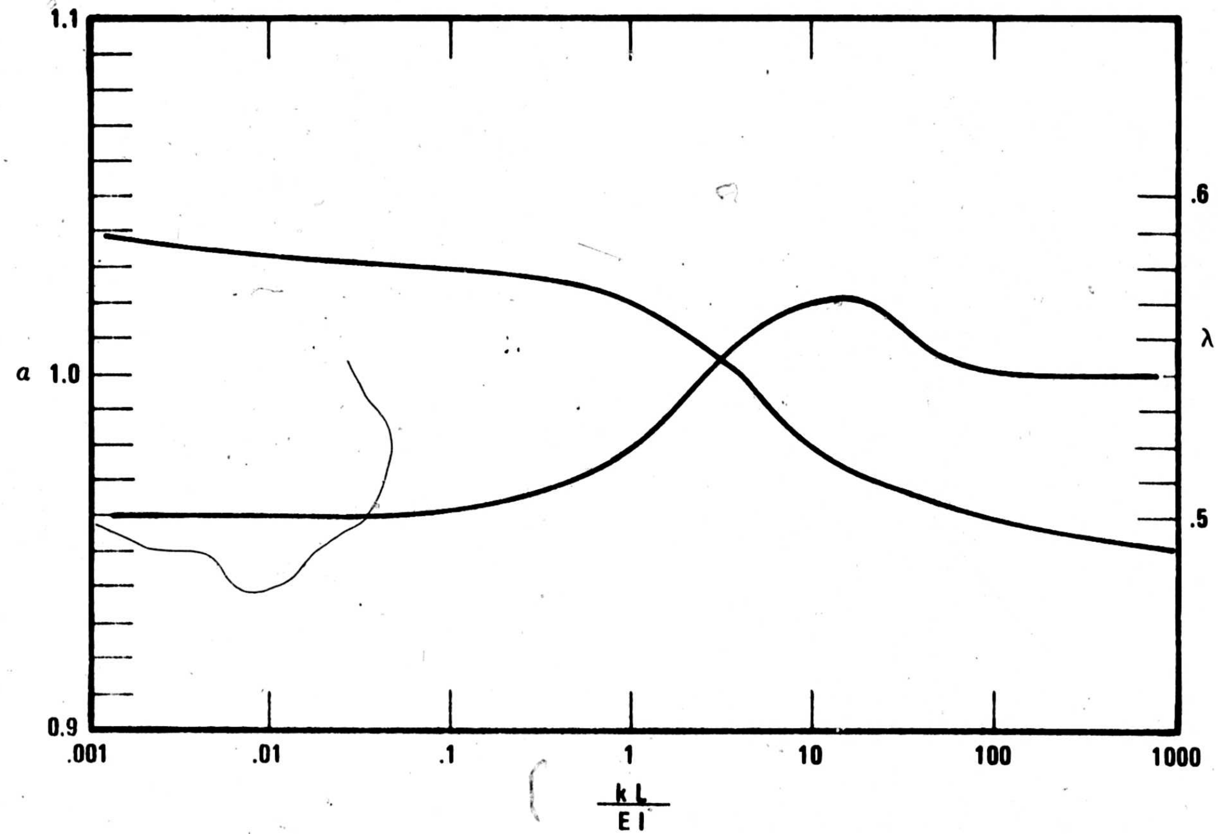

Figure 4. A Plot of the Error Factor, α, and Maximum Flexibility Point, λ, Against Rotational End Restraint at the Pinned End of a Clamped-Pinned Beam (Figure 1b).

INTERMEDIATE LATERAL SUPPORT AND HIGHER MODES

The consistency between the many cases studied leads to the prospect that a general but approximate relationship might exist between the critical instability load and the flexibility coefficient. However, the formulation of a general analytic procedure from which such results can be deduced appears somewhat more complex at present; therefore, it seems more efficient to study the problem further by means of specific examples. Quite often, in practice, a column will be supported laterally at one or more points, such a configuration being adopted generally to improve the load-carrying ability. In some instances, the intermediate supports can be considered as a perfectly pinned or clamped restraint, in which case the column may be subdivided into the intervals between supports, which in turn may be taken as columns with applied end moments. For many other practical cases, though, the intermediate supports are far from ideal and, for purposes of analysis, can be represented by rotational and lateral springs located at discrete points. The spring stiffness, then, can be specified to reflect closely the stiffnesses of real supports. An example of such a construction is depicted in Figure 1d, where a pinned column is subjected to a lateral spring restraint at its midpoint. The critical compressive axial load for symmetric deflection is determined when the spring reaction tends to infinity and is defined by

$$\sin u \left[- \sin u + u\left(1 - 16u^2 \frac{EI}{kL^3}\right) \cos u \right] = 0 \tag{29}$$

where

$$P_{cr} = 4u^2 \frac{EI}{L^2} \tag{30}$$

Now, when $k = \infty$ (rigid support), the lowest nontrivial root is $u = \pi$, from which

$$P_{cr} = 4\pi^2 \frac{EI}{L^2} \tag{31}$$

and when $k = 0$ (no support), the lowest root is $u = \pi/2$, from which

$$P_{cr} = \pi^2 \frac{EI}{L^2} \tag{32}$$

These are clearly the cases when two waves and one wave are produced, and they establish limits on u. As k is increased from zero, the lowest root of equation (29) corresponding to symmetric buckling is determined when the bracketed term vanishes. This happens only so long as

$$1 - 16u^2 \frac{EI}{kL^3} \leq 0 \tag{33}$$

or

$$P \geq \frac{kL^3}{16\pi^2 EI} \quad \frac{4\pi^2 EI}{L^2} \tag{34}$$

If this value is larger than P_{cr} for a rigid support, then the column must buckle with two waves. The transition from one to two waves occurs, then, when

$$P = (P_{cr}) \text{ for two waves} \tag{35}$$

or when

$$k = 16\pi^2 \frac{EI}{L^3} \tag{36}$$

For k greater than this value, antisymmetric buckling occurs as defined by equation (30). For k less than this value, symmetric single-wave buckling occurs at a load when (see equation (29))

$$- \sin u + u(1 - 16u^2 \frac{EI}{kL^3}) \cos u = 0 \tag{37}$$

The solution can be represented approximately as

$$P_{cr} = (1 + \frac{3\ k\ L^3}{16\pi^2\ EI}) \frac{\pi^2 EI}{L^2} \tag{38}$$

When the spring stiffness is zero or less than that defined in equation (39), the point of minimum lateral stiffness is clearly the midpoint. The flexibility coefficient as shown in Appendix IV is then

$$f = \frac{L^3}{48\ EI} \left[\frac{1}{1 + \frac{kL^3}{48\ EI}} \right] \tag{39}$$

Using equation ( ), there results

$$P_{cr} f = \alpha \frac{\pi^2 L}{48} \tag{40}$$

where

$$\alpha = \frac{1 + \dfrac{3\ kL^3}{16\pi^2\ EI}}{1 + \dfrac{kL^3}{48\ EI}} \tag{41}$$

The factor α is then a measure of the deviation of Pf from the constant value and is plotted versus k in Figure 5. It is evident that the maximum error is less than 7% and occurs at the transition from the first to the second buckle mode.

Now, when $\frac{kL^3}{EI} > 16\pi^2$, there is an inflection point at L/2, and minimum lateral stiffness clearly does not occur here. It does occur, however, at two points located midway between the ends and center support. This requires some modification of our previous results. Consider the simultaneous application of a point load, Q, at one of the points and an equal and oppositely directed load at the other. The sum of the resulting deflections is

$$\delta = \delta_1 + \delta_2 = \frac{QL^3}{96\ EI} \tag{42}$$

and, if we define a new flexibility coefficient

$$\tilde{f}_2 = \frac{\delta}{Q} = \frac{L^3}{192\ EI} \tag{43}$$

then

$$P_{cr} \tilde{f}_2 = \frac{4\pi^2\ EI}{L^2} \ \frac{L^3}{192\ EI} = \frac{\pi^2 L}{48} \tag{44}$$

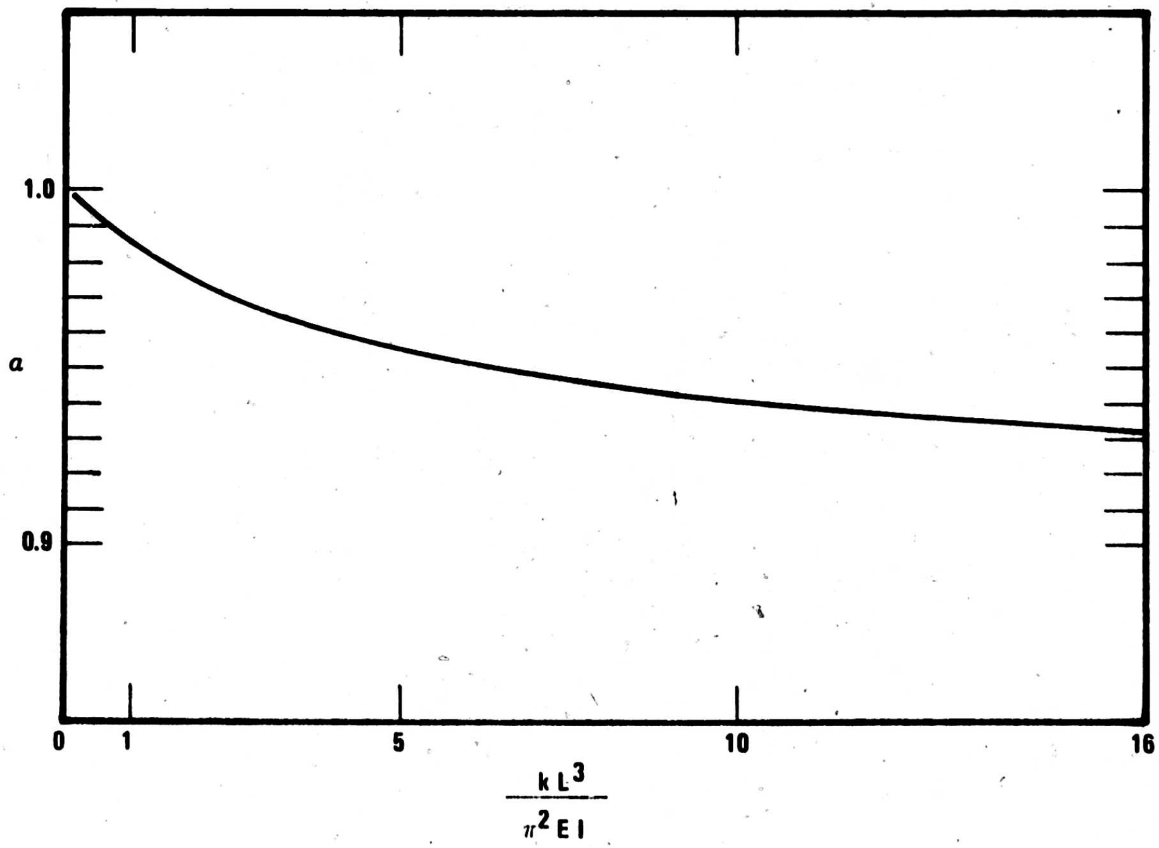

Figure 5. A Plot of the Error Factor, α, Against Lateral Restraint at the Midpoint of a Pinned Beam (Figure 1d).

which is just the same result as that for the earlier examples.

This approach can also be extended to other situations in which higher modes are critical. The first example taken is that of a column that is encastre at both ends and pinned at a center support. In this case, the buckle mode is again antisymmetric and, using equations (16) and (17), the points of minimum stiffness occur at $X = 0.29\ L$ and $0.71\ L$. If oppositely directed loads, Q, are applied at these points, then it follows in a similar manner to the previous case:

$$\delta = \delta_1 + \delta_2 = 2\ (.00984)\ \frac{Q\left[\frac{L}{2}\right]^3}{EI} \tag{45}$$

and

$$\tilde{f}_2 \doteq \frac{\delta}{Q} = 0.00246\ \frac{L^3}{EI} \tag{46}$$

The critical compressive load is

$$P_{cr} = 2.05\pi^2\ \frac{EI}{(\frac{L}{2})^2} = 8.2\pi^2\ \frac{EI}{L^2} \tag{47}$$

and so

$$P_{cr}\tilde{f}_2 = .97\ \frac{\pi^2 L}{48} \tag{48}$$

giving an error of 3%.

If we consider buckling in the third mode for a pinned-ended column, then one can show that three alternating forces are required and that if

$$\tilde{f}_3 = \frac{\delta_1 + \delta_2 + \delta_3}{Q} \tag{49}$$

then

$$P_{cr}f = \frac{\pi^2 L}{48} \tag{50}$$

with zero resulting error.

It is possible to treat higher mode buckling cases with the various boundary conditions discussed earlier by subdividing the column at inflection points into a series of column elements. As before, the flexibility coefficient must be defined by applying forces to each element at points of minimum stiffness and summing the magnitudes of the deflections produced; then

$$\tilde{f}_n \doteq \frac{\sum^{n} \delta_i}{Q} \tag{51}$$

and

$$P_{cr}\tilde{f} = \frac{\pi^2 L}{48} \tag{52}$$

The $P\tilde{f}$ product is exact for pinned-ended conditions. Since for higher mode buckling with other end conditions, the interior portion of the column deforms into sinusoidal waves, it can be concluded that the error for these cases diminishes with increasing buckle mode.

These results suggest a generalized procedure for situations when the buckling load corresponds to higher modes. The maximum generalized flexibility coefficient, $\tilde{f}_n = \sum^{n} \delta_i/Q$, must be determined for all contemplated modes by first applying a single concentrated lateral load and then adding additional loads in an alternating fashion, in each case adjusting the positions of the loads so that the deflection sum is a maximum. This defines the "maximum flexibility" configuration for that mode, and the mode with the lowest value will then be preferred, so that

$$P_{cr}\tilde{f}_n \simeq \frac{\pi^2 L}{48} \tag{53}$$

LATERAL RESTRAINT AT ONE END

We have illustrated by the several examples that when the end fixity conditions as regards rotational restraint are varied, the product of the critical load and the influence coefficient at the point of maximum compliance is very nearly constant for both uniform and nonuniform columns. We now turn to investigate the influence of variation in lateral restraint at one end. To study this, we consider the column that is built in at its root but is free at the tip. The critical load for such a column when EI is constant along the length is given by

$$P_{cr} = \frac{\pi^2 EI}{4L^2} \tag{54}$$

The point of maximum compliance is clearly the tip of the column, and the flexibility coefficient is from the well-established deflection formula given by

$$f = \frac{L^3}{3\,EI} \tag{55}$$

Thus, the product

$$P_{cr}f = \frac{\pi^2 EI}{4L^2}\,\frac{L^3}{3\,EI} = \frac{\pi^2 L}{12\,EI} \tag{56}$$

There is clearly a large discrepancy here. It is interesting, therefore, to determine at what point along the column the flexibility coefficient must be determined for this product to have the same value as for the other cases considered. It follows that, since the coefficient varies with the cube of the distance of the load point from the root of the column, the corresponding point is defined by the length L_1, where

$$L_1 = \frac{5L}{8} \simeq .625L \tag{57}$$

It is, perhaps, worthwhile to note that this point is not far removed from that which is determined for the case when the tip is pinned. In that case, the point of maximum compliance was located at .586 L.

To what degree this near correspondence would be influenced by a reasonable nonuniformity of section is now studied. The case considered here is that of a cantilevered column in which the cross-sectional characteristic is defined by

$$EI = EI_o \left(1 - \frac{1}{5}\frac{x^2}{L^2}\right) \tag{58}$$

where EI_o is the value of the section parameter at the clamped end and x is measured from that point. The critical load for this structure is given by[17]

$$P_{cr} = \frac{12\ EI_o}{5L^2} \tag{59}$$

The deflection due to a unit load (the flexibility coefficient) positioned at a point λL from the fixed end is given by

$$f = \int_o^{\lambda L} \frac{(\lambda L - x)^2}{EI_o \left[1 - \frac{1}{5}\left(\frac{x}{L}\right)^2\right]}\, dx \tag{60}$$

The point is to determine the value of λ which makes the product $P_{cr}f$ have the value previously established. For this condition to be met, the value of f must be given by

$$f = \frac{5\pi^2}{(12)\ (48)} \cdot \frac{L^3}{EI_o} \tag{61}$$

A solution of equations (59) and (60) yields the value

$$\lambda = 0.634 \tag{62}$$

or

$$L_1 = 0.634L \tag{63}$$

which is remarkably close to the value 0.625 obtained for the uniform cantilever, equation (56).

I+ appears, then, for cantilever-type configurations of uniform or non-uniform geometry, that the product $P_{cr}f$ can be taken as constant with reasonable error, provided the flexibility coefficient is measured within 13% of the column midpoint. When the maximum value of the coefficient falls outside this limit, then, the value at the limit must be used instead.

CONCLUSIONS

The research outlined in this paper has established a practical yet simple method of determining the end fixity of realistic columns. It has been shown, both analytically and experimentally, that the end fixity coefficient for a column can be obtained within usual engineering tolerance by dividing the parameter $\pi^2L/48$ by the product of the Euler load and the maximum value of the flexibility coefficient for a point lateral load applied between 3/8 and 5/8 of the span of the column. This rule is applicable for all conditions of restraint, at either end of the column, so long as the applied force is constrained to remain vertical.

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APPENDIX I

FLEXIBILITY COEFFICIENT FOR A TAPERED BEAM

Consider the clamped-pinned beam shown in Figure 1b. The section stiffness EI is variable and is given by

$$EI = EI_o \left(1-a^2\xi^2\right) \tag{64}$$

where $\xi = x/L$ and where a is the taper ratio. It is desired to find the position λ for which the flexibility coefficient associated with loads and displacements at $x = \lambda L$ is maximized. The procedure employed here is the well-known dummy load method.

The first requisite in the analysis is to obtain the actual moment distribution, for which it is necessary to determine the reaction force, R, at the end of the beam. Toward this objective, the internal complementary work is given by

$$U^* = \int_0^L \frac{(M_1)^2}{2EI}\, dx \tag{65}$$

Since the reactions do no work, the complementary potential is given by an expression identical to equation (65). In terms of the unknown redundant R, the moment distribution is given by

$$M_1 = \begin{cases} RL\,(1-\xi) - QL\,(\lambda-\xi) & 0 \le \xi \le \lambda \\ RL\,(1-\xi) & \lambda \le \xi \le 1 \end{cases} \tag{66}$$

Substituting equation (66) and equation (64) into equation (65) gives, after rearrangement,

$$U^* = \frac{L^3}{2\,EI_o}\left\{ R^2 \int_0^1 \frac{(1-\xi)^2 d\xi}{1-a^2\xi^2} - 2\,RQ \int_0^\lambda \frac{(1-\xi)(\lambda-\xi)}{1-a^2\xi^2}\, d\xi \right.$$

$$\left. + Q^2 \int_0^\lambda \frac{(\lambda-\xi)^2}{1-a^2\xi^2}\, d\xi \right\} \tag{67}$$

Now, the force at the end is that R, among all R's in equilibrium with the later load Q, which permits compatible deformations of the structure; equivalently,

$$\frac{\partial U^*}{\partial R} = 0 \tag{68}$$

Carrying out the indicated differentiation of equation (67) and solving for R, we have

$$(R/Q) = \frac{\int_0^\lambda \frac{(1-\xi)(\lambda-\xi)}{1-a^2\xi^2} d\xi}{\int_0^1 \frac{(1-\xi)^2 d\xi}{1-a^2\xi^2}} \tag{69}$$

The deflection, δ, of the beam is given by the dummy load method as

$$\delta = \int_0^L \frac{M_o M_1 dx}{EI} \tag{70}$$

where M_o is the moment distribution due to the dummy load. In this case, the dummy load is applied at the same place as the actual load, Q, so

$$M_o(x) = \frac{1}{Q} M_1(x) \tag{71}$$

Substitution into equation (70) gives, after rearrangement,

$$(\delta/Q) = \frac{2}{Q^2} U^* \tag{72}$$

Thus

$$\frac{1}{2}\,(\delta/Q) = \frac{L^3}{2\,EI_o}\left\{\int_0^{\lambda} \frac{[(R/Q)\,(1-\xi) - \lambda + \xi\,]^2}{1 - a^2\xi^2}\, d\xi \right.$$

$$\left. + \,(R/Q)^2 \int_{\lambda}^{1} \frac{(1-\xi)^2}{1 - a^2\xi^2}\, d\xi \right\} \tag{73}$$

Now, equation (69) can be integrated in closed form to yield

$$R_1(^R/Q) = a^2\lambda \,\ln\,(\frac{1 + a\lambda}{1 - a\lambda}) + 2a\,(1 + \lambda)\,\ln\,\sqrt{1 - a^2\lambda^2}$$

$$+ \,2\,\ln\,\frac{1 + a\lambda}{\sqrt{1 - a^2\lambda^2}} - 2\,a\lambda \tag{74}$$

where

$$R_1 = a^2\,\ln\frac{1 + a}{1 - a} + 4a\,\ln\sqrt{1 - a^2} + 2\,\ln\frac{1 + a}{\sqrt{1 - a^2}} - 2a \tag{75}$$

Furthermore, substitution of equations (74) and (75) into equation (73) gives, after considerable simplification,

$$2a^3\,\frac{(^\delta/Q)}{(L^3/EI_o)} = -\,R_1(^R/Q)^2 + a^2\lambda^2\,\ln\,\frac{1 + a\lambda}{1 - a\lambda}$$

$$+ \,4a\lambda\,\ln\sqrt{1 - a^2\lambda^2} + 2\,\ln\,\frac{1 + a\lambda}{\sqrt{1 - a^2\lambda^2}} - 2a\lambda \tag{76}$$

Since it was desired to maximize the influence coefficient, it was necessary to differentiate equations (74) and (76). Carrying out the differentiation yields, after simplification,

$$R_1\,\frac{d}{d\lambda}\,(^R/Q) = a^2\,\ln\,(\frac{1 + a\lambda}{1 - a\lambda}) + 2a\,\ln\,\sqrt{1-a^2\lambda^2} \tag{77}$$

or

$$a^2 \frac{d}{d\lambda} \left\{ \frac{(\delta/Q)}{(L^3/EI_o)} \right\} = a \left[\lambda - (R/Q) \right] \ln \left(\frac{1 + a\lambda}{1 - a\lambda} \right)$$

$$+ 2 \left[1 - (R/Q) \right] \ln \sqrt{1 - a^2\lambda^2} \qquad (78)$$

Hence, the problem of maximizing the influence coefficient is expressed by

$$f(\lambda) = a^2 \frac{d}{d\lambda} \left\{ \frac{(\delta/Q)}{(L^3/EI_o)} \right\} = 0 \qquad (79)$$

Again, it is apparent that the root .. cannot be found in closed form; a numerical scheme is necessary.

Newton's method was used here, with the iteration being accomplished via a digital computer. The maximum flexibility coefficient and the associated position for a taper ratio, a, of 1/3 were found to be:

$$\lambda = 0.599 \qquad (80)$$

$$f = 1.09 \times 10^{-2} \qquad (81)$$

Taper Ratio	Maximizing Lambda	Pinned-End Reaction	Flexibility Coefficient ($\times 10^2$)
.1	.589456	.416455	1.01000
.2	.593348	.418838	1.04138
.3	.597658	.421547	1.07608
.4	.602422	.4246	1.11483
.5	.607749	.428089	1.15850
.6	.613806	.432163	1.20853
.7	.620837	.437029	1.26699
.8	.629294	.443103	1.33743
.9	.640046	.451192	1.42667

APPENDIX II

FLEXIBILITY COEFFICIENT FOR A BEAM WITH EQUAL ROTATIONAL END RESTRAINT

The problem considered is to determine the maximum flexibility coefficient of a pinned beam with rotational restraint as shown in Figure 1c. A complementary energy approach similar to that of Appendix I was used. The problem is symmetric about the beam midpoint, which is clearly the point of maximum flexibility coefficient. The moment distribution can be written

$$M = M_o + \frac{Qx}{2} \quad 0 \le x \le L/2 \tag{82}$$

so the internal complementary energy is

$$U^* = \frac{1}{2\,EI}\int_0^L M^2 dx = \frac{L}{8\,EI}\left(4M_o^{\,2} + M_o QL + \frac{Q^2L^2}{12}\right) \tag{83}$$

The work done by the spring reaction forces is $-M_o\theta$, where θ is the end rotation. The total complementary potential is then the sum

$$\Pi^* = U^* - M_o\theta \quad . \tag{84}$$

The proper value of M_o acting in the structure is determined by the principle of minimum total complementary potential:

$$0 = \frac{\partial \Pi^*}{\partial M_o} \quad . \tag{85}$$

Using equation (84) in equation (85) and solving for M_o yields

$$M_o = \frac{-QL}{8\left(1 + \frac{EI}{kL}\right)} \tag{86}$$

The midpoint deflection can now easily be found either by the dummy load method or equivalently by Castigliano's First Theorem:

$$\delta = \frac{\partial U}{\partial Q} \tag{87}$$

Using the latter approach requires determination of the strain energy U:

$$U = \frac{L}{8\ EI} \left(4M_o^2 + M_oQL + \frac{Q^2L^2}{12}\right) + \frac{M_o^2}{2k} \tag{88}$$

Now, using equations (86) and (88) in equation (87) gives, after some manipulation,

$$\delta = \frac{QL^3}{48\ EI} \left[1 - \frac{1}{4(1 + \frac{EI}{kL})}\right] \tag{89}$$

at the beam midpoint, from which the flexibility coefficient is determined as

$$f = \frac{\delta}{Q} = \frac{L^3}{48\ EI} \left[1 - \frac{1}{4(1 + \frac{EI}{kL})}\right] \tag{90}$$

APPENDIX III
FLEXIBILITY COEFFICIENT FOR A PINNED BEAM WITH ROTATIONAL RESTRAINT AT ONE END

The maximum flexibility coefficient for the pin-ended beam with rotational restraint at one end (Figure 1c with the leftmost spring removed) can be determined with good accuracy by a Rayleigh-Ritz procedure. This method consists of choosing an approximate function for the deflected shape of the beam and then adjusting the unspecified parameters in the function by minimizing the total potential energy in the system.

For the present case, the deflection function

$$W = \sum_{m=1}^{N} A_m \sin \frac{m\pi x}{L} \tag{91}$$

consisting of the first N sin terms was chosen. The total potential energy can then be written

$$\Pi = \frac{EI}{2} \int_0^L \left(\frac{\partial^2 W}{\partial x^2} \right)^2 dx + \frac{k}{2} \left[\frac{\partial W(L)}{\partial x} \right]^2 - Q\, W(\lambda L) \tag{92}$$

The proper values of the constants, A_i, are determined by requiring that Π assume a minimum value; i.e.,

$$0 = \frac{\partial \Pi}{\partial A_i} \quad \text{for } i = 1,2 \ldots\ldots N \tag{93}$$

Substituting equation (91) in equation (92) and carrying out the indicated operations yields an algebraic equation. Using this in equation (93), then, gives a set of simultaneous linear equations determining the constants, A_m, as follows:

$$(m\pi)^4 A_m + 2 \frac{kL}{EI} m\pi \cos (m\pi) \left[\sum_{n=1} A_n \, n\pi \cos (n\pi) \right]$$

$$= 2 \frac{QL^3}{EI} \sin (\lambda m\pi) \text{ for } m = 1,2 \ldots\ldots N \tag{94}$$

These equations can be solved in a straightforward manner for the constants, A_m, given a value, λ, for the load position. For the present case, a direct numerical solution was completed and the resulting coefficients, A_m, were used to find the flexibility coefficient at the point λL:

$$f = \frac{\delta}{Q} = \frac{W(\lambda L)}{Q} \tag{95}$$

Values of λ were adjusted in an iteration scheme until the flexibility coefficient assumed a maximum value. The results are presented in Figure 3.

APPENDIX IV
FLEXIBILITY COEFFICIENT FOR A PINNED BEAM WITH INTERMEDIATE LATERAL SUPPORT

The maximum flexibility coefficient for the pinned beam shown in Figure 1d with a lateral spring support at its midpoint can be determined in a direct manner by energy methods. For values of spring stiffness less than that given in equation (36), i. e.,

$$k < 16\pi^2 \frac{EI}{L^3} \tag{96}$$

the point of maximum flexibility coefficient (minimum stiffness) is the midpoint. If a load, Q, is applied here, the moment distribution is

$$\begin{aligned} M &= \frac{Q - R}{2} x \qquad 0 \leq x \leq L/2 \\ &= \frac{Q - R}{2} x - (Q - R)(x - L/2) \quad \frac{L}{2} \leq x \leq L \end{aligned} \tag{97}$$

where R is the upward reaction due to the spring. Essentially, the same energy procedure outlined in Appendixes I and II is used here. The total complementary potential can be written

$$\pi^* = \frac{1}{EI} \int_0^{L/2} \left(\frac{Q - R}{2} x\right)^2 dx - R(-\delta) \tag{98}$$

where δ is the midpoint deflection in the downward direction of Q. As before, the unknown reaction, R, has that value which minimizes π^*; equivalently,

$$0 = \frac{\partial \pi^*}{\partial R} \tag{99}$$

Using equation (98) in equation (99) and simplifying gives the result

$$R = \frac{Q}{1 + \frac{48\ EI}{kL^3}} \tag{100}$$

The deflection of the spring is just the deflection at the midpoint due to Q. Thus,

$$d = \frac{R}{k} = \frac{QL^3}{48\ EI}\left[\frac{1}{1 + \frac{kL^3}{48\ EI}}\right] \tag{101}$$

and the flexibility coefficient is given by

$$f = \frac{L^3}{48\ EI}\left[\frac{1}{1 + \frac{kL^3}{48\ EI}}\right] \tag{102}$$

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13. ABSTRACT

The question of column end fixity is critically reconsidered. An effective method for determination is established. This method relies upon the fact that there exists a readily determined point in the span of the column for which the deflection due to a given side force is maximum. The product of the coefficient which defines the flexibility at this point and the Euler load appropriate to the actual end conditions is, to all intents and purposes, constant.

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